An Overview of Barrier Options

working draft – This paper is not a final draft, and remains to be completed

http://www.global-derivatives.com

info@global-derivatives.com

Kevin Cheng

Contents: **Page:**

Introduction

Barrier options are path-dependent options which come in various flavours and forms, but their key characteristic is that these types of options are either initiated or exterminated upon reaching a certain barrier level; that is, they are either knocked in or knocked out.

With a wide diversity within the barrier option category, in this overview we consider the most basic type of barrier option - the single barrier.

This option comes in 8 flavours, each with its own characteristics, which we outline below:

Types:

1. Up & In
2. Up & Out
3. Down & In
4. Down & Out

Where each type of barrier can take the form of either a call or a put - giving us a total of 8 single barrier types. An "In" barrier means that a barrier becomes active once crossing a particular barrier level; for example, an Up & In barrier becomes active when the underlying price hits a barrier from below.

Barriers also come in various other forms including double barriers, parisians, and partial time barriers, which can all eventually be found in their respective pages. Here, we discuss the various methods of pricing single barriers.

Barriers can take either American or European forms, and despite its seemingly complex payoffs, they are widely used in the markets and are generally cheaper than plain vanilla type options.

Rebates

Rebates are pre-defined payoffs which are sometimes given when a barrier is expires worthless, and although barriers with rebates are not traded as much as barriers without, we will consider rebates in a later section.

Pricing:

1) Analytical Closed Form (Merton, Reiner & Rubinstein)

Since barrier options were traded on the OTC in the late 60s they have been used extensively to manage risks related to commodities, FX and interest rate exposures.

Merton (1973) provided the first analytical formula for a down and out call option which was followed by the more detailed paper by Reiner & Rubinstein (1991) which provides the formulas for all 8 types of barriers. Haug (1998) gives a generalisation of the set of formulas provided by Reiner & Rubinstein (RR); both of which we will consider.

We consider the 16 single barrier options as follows, given by RR under a Black & Scholes framework.

When the barrier level is lower than or equal to the strike price, the following values are given:

Call Values:

$$c_{up-in} \mid H \leq X = c_{BS}$$

$$c_{up-out} \mid H \leq X = c_{BS} = (c_{up-in} \mid H > X)$$

$$c_{down-in} \mid H \leq X = Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(y) - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N(y - \sigma\sqrt{T})$$

$$c_{down-out} \mid H \leq X = c_{BS} - (c_{down-in} \mid H \leq X)$$

Put Values:

$$\begin{aligned} p_{up-out} \mid H \leq X = &-Se^{-DT} N(-x_1) + Xe^{-rT} N(-x_1 + \sigma\sqrt{T}) + \\ &Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(-y_1) - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N(-y_1 + \sigma\sqrt{T}) \end{aligned}$$

$$p_{down-out} \mid H \leq X = p_{BS} - (p_{down-in} \mid H \leq X)$$

$$p_{down-in} \mid H \le X = -Se^{-DT} N(-x_1) + Xe^{-rT} N(-x_1 + \sigma\sqrt{T}) +$$
$$Se^{-DT}\left(\frac{H}{S}\right)^{2m}\left[N(y) - N(y_1)\right] - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}\left[N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T})\right]$$

$$p_{up-in} \mid H \le X = p_{BS} - (p_{up-out} \mid H \le X)$$

When the barrier is greater than the strike price, the following formulae are used:

Call Values:

$$c_{up-out} \mid H > X = c_{BS}$$

$$c_{down-out} \mid H > X = Se^{-DT} N(x_1) - Xe^{-rT} N(x_1 - \sigma\sqrt{T}) -$$
$$Se^{-DT}\left(\frac{H}{S}\right)^{2m}\left[N(y_1)\right] + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N(y_1 - \sigma\sqrt{T})$$

$$c_{down-in} \mid H > X = c_{BS} - (c_{down-out} \mid H > X)$$

$$c_{up-in} \mid H > X = Se^{-DT} N(x_1) - Xe^{-rT} N(x_1 - \sigma\sqrt{T}) -$$
$$Se^{-DT}\left(\frac{H}{S}\right)^{2m}\left[N(-y) - N(-y_1)\right] +$$
$$Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}\left[N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})\right]$$

Put Values:

$$p_{down-in} \mid H > X = p_{BS}$$

$$p_{down-out} \mid H > X = 0$$

$$p_{up-in} \mid H > X = -Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(-y) + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N(-y + \sigma\sqrt{T})$$

$$p_{up-out} \mid H > X = p_{BS} - (p_{up-in} \mid H > X)$$

Where:

$$y = \frac{\ln\left(H^2 / SX\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T} \qquad m = \frac{r - D + 0.5\sigma^2}{\sigma^2}$$

$$x_1 = \frac{\ln\left(S / H\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T} \qquad y_1 = \frac{\ln\left(H / S\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}$$

Where N(x) is the cumulative normal distribution of x, H is the barrier value, S is the asset price, X is the strike price, c_{BS} and p_{BS} are the values of a European call and put option respectively under the Black-Scholes framework.

Haug handles the equations differently by using a set of standard equations, and making use of binary variables in order to generalise the formula for implementation. Both Haug & RR formulae are the same, but merely presented differently; the choice of use if up to the programmer, but we use Haug's method for more versatile programming.

Haug gives 6 standard formulas as follows: - We use similar notation as Haug's text for generalisation.

$$A = \phi S e^{-DT} N(\phi x_1) - \phi X e^{-rT} N(\phi x_1 - \phi\sigma\sqrt{T})$$

$$B = \phi S e^{-DT} N(\phi x_2) - \phi X e^{-rT} N(\phi x_2 - \phi\sigma\sqrt{T})$$

$$C = \phi S e^{-DT} \left(\frac{H}{S}\right)^{2(m+1)} N(\eta y_1) - \phi X e^{-rT} \left(\frac{H}{S}\right)^{2m} N(\eta y_1 - \eta\sigma\sqrt{T})$$

$$D = \phi S e^{-DT} \left(\frac{H}{S}\right)^{2(m+1)} N(\eta y_2) - \phi X e^{-rT} \left(\frac{H}{S}\right)^{2m} N(\eta y_2 - \eta\sigma\sqrt{T})$$

$$E = Ke^{-rT}\left[N(\eta x_2 - \eta\sigma\sqrt{T}) - \left(\frac{H}{S}\right)^{2m} N(\eta y_2 - \eta\sigma\sqrt{T})\right]$$

$$F = K\left[\left(\frac{H}{S}\right)^{m+\lambda} N(\eta z) + \left(\frac{H}{S}\right)^{m-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T})\right]$$

Where:

$\lambda = \sqrt{m^2 + \frac{2(r-D)}{\sigma^2}}$	$m = \frac{r - D - 0.5\sigma^2}{\sigma^2}$
$x_1 = \frac{\ln\left(S/X\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$	$x_2 = \frac{\ln\left(S/H\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$
$y_1 = \frac{\ln\left(H^2/SX\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$	$y_2 = \frac{\ln\left(H/S\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$

$$z = \frac{\ln\left(H/S\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

K is the rebate which is paid out if the barrier has not been knocked in during its life. Haug then gives the values of each of the barrier options and their respective binary values.

If X < H	Value	η	ϕ
Down-In Call	A-B+D+E	1	1
Down-Out Call	B-D+F	1	1
Up-In Call	B-C+D+E	-1	1
Up-Out Call	A-B+C-D+F	-1	1
Down-In Put	A+E	1	-1
Down-Out Put	F	1	-1
Up-In Put	C+E	-1	-1
Up-Out Put	A-C+F	-1	-1
If X > H	**Value**	η	ϕ
Down-In Call	C+E	1	1
Down-Out Call	A-C+F	1	1
Up-In Call	A+E	-1	1
Up-Out Call	F	-1	1
Down-In Put	B-C+D+E	1	-1
Down-Out Put	A-B+C-D+F	1	-1
Up-In Put	A-B+D+E	-1	-1
Up-Out Put	B-D+F	-1	-1

2) Continuity Correction (Broadie, Glasserman, Kou)

The aforementioned analytic formulas present a method to price barrier options in continuous time, but often in industry, the asset price is sampled at discrete times, where periodic measurements rather than a continuous lognormal distribution of the asset prices is assumed.

Broadie, Glasserman & Kou arise at an adjustment to the barrier value to account for discrete sampling as follows:

$$H = He^{\alpha\sigma\sqrt{T/m}}$$

For "up" options which hit the barrier from underneath, the value of α is 0.5826, whereas for "down" options where the barrier is hit from the top, the value of α is -0.5826, where m is the number of times the asset price is sampled over the period.

3) Binomial Method

Like most other path-dependent options, barrier options can be priced via lattice tree such as binomial, trinomial or adaptive mesh models by solving the PDE using a generalised finite difference method. The classic Cox, Ross & Rubinstein (1979) paper introduces the binomial method which has since been adapted to various other option types including barrier options.

The up and down movements of barrier options under the binomial method is the same as it is for other types of options, and we reiterate these as being:

$$d = e^{-\sigma\sqrt{T}} \qquad\qquad u = e^{\sigma\sqrt{T}}$$

Which can be reduced to:

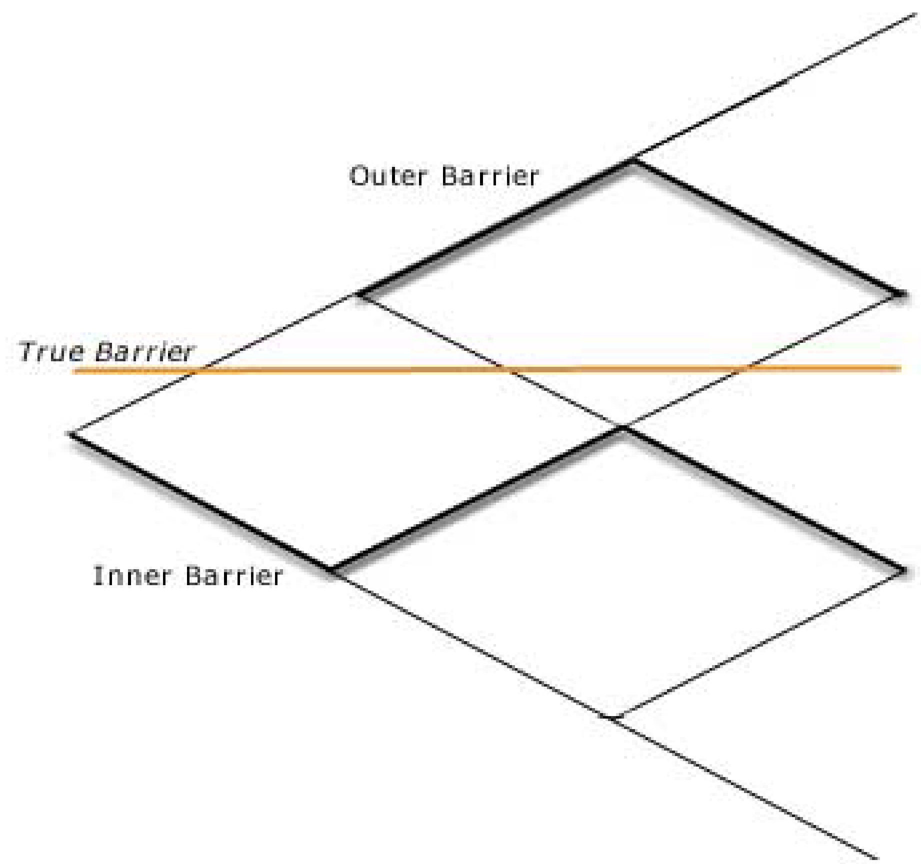
$$d = \frac{1}{u}$$

and the probabilities of an up and down movements are respectively given as:

$$p = \frac{e^{(r-D)T} - d}{u - d}$$

Where the capital D denotes the dividend yield, lowercase d represents the down magnitude and probability of a down movement = 1 - p.

An important aspect to note is that with the existence of barriers, the 'zig-zag' movements of a binomial model will undoubtedly create problems, as the true barrier of the barrier in question is often not the same barrier implied by the tree. Derman et al (1995) and Hull (2002) illustrate this by considering an inner barrier and an outer barrier, between which lies the true barrier.



The existence of an inner and outer barrier are due to the implicit assumption of the binomial tree which places the barriers at node points, where in fact, the true barrier usually lies in between.

Hull (2002) and Derman et al. (1995) present several alternatives to adjust for the discrepancy with using lattice methods:

1) By positioning nodes where the barriers are.
- This approach is effective unless the initial asset price is close to the barrier

2) Adjust for the nodes by assuming the barrier calculated by the tree is incorrect.
- By calculating the inner and outer barriers and working backwards through the lattice, the first value which is encountered (i.e. the inner barrier) is given as correct, followed by the second value encountered (the outer barrier), and so forth. Once we reach the initial node, we can interpolate between the values (see Hull).

3) Finally, we can use an adaptive mesh model; which we introduce shortly.

Alternatively, Boyle & Lau (1994) suggest a way to determine which nodes will give good approximate pricing of a barrier option (where the inner and outer nodes are closest together). The number of time steps which will give the most accurate prices when using the binomial lattice is given as:

$$Node(i) = \frac{i^2 \sigma^2 T}{\left[\ln\left(S/H\right)\right]^2}$$

Where i is equal to 1, 2, 3...,n nodes. For example, if the stock price was 100, the barrier was set at 105, volatility is 20% and time to maturity is 1 year, then the corresponding optimal node to a normal node of 1 (i) would be:

$$Node(1) = \frac{1^2 0.20^2 1}{\left[\ln\left(100/105\right)\right]^2}$$

Which gives us a value of approximately 17; this being an 'optimal node'.

Nonetheless, although these optimal nodes will provide a more accurate value for the barrier, a large number of time steps should be used to determine a reasonable value. This section will soon incorporate a detailed illustration of how convergence of barrier options under the binomial method often takes thousands of time steps for values to be considered as reasonable, but Derman et al (1995) show that convergence is slow due to two sources of error for binomial methods in general:

1. Stock Price Quantisation Error

Where inaccuracy is caused by the assumption of continuous observations of the stock price

2. Option Specification Error

The lattice itself does not accurately represent the option in question, in that adjustments to the lattice results in a different representation to that of the real option.

It is then shown that these two sources of error, as pronounced as they are when valuing simple vanilla options under the binomial tree, presents much more difficult problems and even slower convergence.

We can also extend the binomial tree to an implied binomial tree to incorporate the volatility smile (see Derman & Kani - 1994)

4) Trinomial Method

The trinomial method is essentially a simple and elegant extension to the binomial method, and in pricing plain vanilla options, converges quickly compared to the binomial method. In the case of barrier options, the standard trinomial method suggested by Boyle (1986) can be applied, but again, the lattice errors significantly slow down convergence.

Ritchken & Kamrad (RK) (1991) propose a modified technique to value barrier options under a trinomial framework, which gives better results than the standard lattice.

They give the up, down and middle magnitudes as:

$$u = \lambda\sigma\sqrt{T} \qquad d = -\lambda\sigma\sqrt{T} \qquad m = 0$$

With the respective probabilities as:

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{T}}{2\lambda\sigma} \qquad p_d = \frac{1}{2\lambda^2} - \frac{\mu\sqrt{T}}{2\lambda\sigma} \qquad p_m = 1 - \frac{1}{\lambda^2}$$

the drift term μ is given as:

$$\mu = r - D - 0.5\sigma^2$$

and λ is the control term which governs determines the space between the price layers on the trinomial tree. When λ is equal to 1, the trinomial tree becomes the well known binomial tree, and when λ is equal to $\sqrt{3}$, the RK trinomial tree becomes the standard trinomial tree given by Boyle. To determine the value of λ to use, Levitan (2001) uses a method which makes use of the number of consecutive down movements leading to the lowest lattice on the tree. The formula for λ is given to be:

$$\lambda = \frac{\ln\left(S/H\right)}{n_0 \sigma \sqrt{T}}$$

Where n_0 is the number of consecutive down movements leading to the lowest lattice on the tree. This is shown to converge rapidly towards the actual value. Illustrations of this will follow soon.

Note that the trinomial method is essentially equivalent to pricing under a finite difference method. (see Rubinstein 2000)

5) Barrier Options under Jump Diffusion

The jump diffusion process is easily implemented within a vanilla option pricing framework as first suggested in Merton's (1976) seminal paper. In pricing barrier options, the assumptions which underlie the jump process require adjustments to incorporate the boundary condition implied by the barrier option.

Research on jump processes for barrier options are still undergoing, and notable work includes that of Leisen (1998) who discretises the asset space rather than the time space and incorporates jump risk into the model.

Beaglehole et al (1999) and later, Metwally & Atiya (M&A) (2002) make use of a Brownian bridge to limit the effects of high-dimensional components under Monte Carlo simulation. This bears some similarities to the model suggested by Leisen, but a main point to note is that M&A discretise with respect to the time space. The use of a bridge greatly increases convergence time and quantitative error, and the authors note that by using this method, the time taken to simulate the price of a barrier option with jumps is about 100 times faster. They also highlight the issue of the boundary crossing problem or first-passage time problem.

6) Other Lattice Methods

Having highlighted the main techniques revolving around the binomial and trinomial models as well as their pitfalls and solutions, we look at alternative lattice methods and several other approaches to the aforementioned binomial and trinomial models.

Broadie, Glasserman & Kou's (1997) continuity correction to barriers makes way for numerous other considerations to discrete pricing of barriers under lattice methods. Further extended by Kou (2001) and Levitan et al (2003), these methods give more realistic pricing methods for barrier options in many cases.

In particular, Levitan et al. (2003) reduce the Cox, Ross & Rubinstein binomial lattice under continuous time to a set of equations to price discrete barrier options under a binomial method. For example, an up & in call is given by:

$$c_{up-in} \mid x^* > m^* = e^{-rT} \sum_{j=\left[\frac{1}{2}(n+x^*)\right]}^{n} \binom{n}{j} p^j (1-p)^{n-j} (Su^j d^{n-j} - X)$$

Where x* and m* are given as:

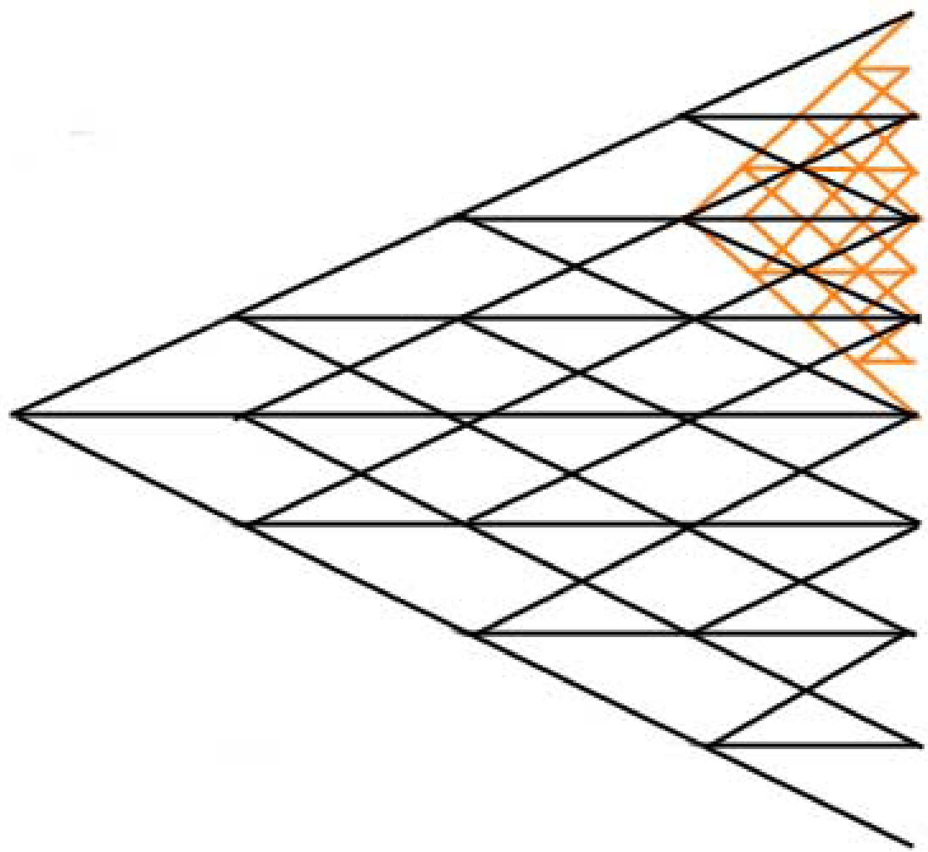
$$x^* = \left[\frac{\ln\left(X/S\right)}{\ln u} \right] \qquad\qquad m^* = \left[\frac{\ln\left(H/S\right)}{\ln u} \right]$$

u is the up magnitude, d is the down magnitude and n is the number of time steps. Similar equations can be drawn for the other options; these will be produced in due course.

7) Adaptive Mesh Model

Figlewski & Gao (1999) proposed a lattice method which builds on the existing binomial or trinomial lattice trees to price American style options. In their subsequent paper, see Figlewski, Gao & Ahn (1999), they extend this approach to price discrete barrier options.

First of all, to illustrate the benefits of the adaptive mesh model, we can draw a lattice which employs the mesh, which is essentially a lattice transposed on top of another lattice. The diagram below shows a generalised adaptive mesh.



The standard trinomial tree given by Boyle (1986) is the lattice in black, and the lattice transposed in orange gives a combined adaptive mesh. (Note that for illustrative simplicity, only the top adaptive mesh is drawn, in fact, the full adaptive mesh would show similar orange grids all along the right hand side of the lattice.

As you can see, the number of paths which the tree can take is greatly increased as we reach exercise, which gives us faster convergence of the tree and generally more accurate results.

In the case of barrier options, the adaptive mesh model can be used, even when the asset price is closed to the barrier, an attribute which creates difficulties in standard lattice models. A mesh is constructed similar to the one shown above, but with the nodes placed along the barriers to give more accurate simulated paths. The barrier option value can then be solved via backwards induction on the tree, and this can be implemented within a Monte Carlo framework.

Figlewski, Gao and Ahn also point out that because of the numerous paths which can be constructed in both binomial and trinomial adaptive meshes, American style barrier options can also be priced in a similar fashion.

We will soon show the improvements in convergence by comparing the adaptive mesh with standard lattice methods.

8) Other Techniques to Price Barriers

With numerous variants of barrier options, we only highlight the various other methodologies to price several types of variations of the barrier option. Detailed analysis for 'exotic barriers' will be found in their respective sections in due course.

For American style barrier options which have no closed form solutions Haug (2000) illustrates the use of analytical solution in which the reflection principle is called upon, and American barrier options are valued based on their relation to plain vanilla options. We can then use analytical approximations to vanilla American options, such as that provided by Barone-Adesi & Whaley (1987) or Bjerksund & Stensland (1993). Haug gives the analytical value of an American down and in barrier, when equal to:

$$c_{Am-down-in}(S,X,H,T,r,D,\sigma) = c_{Am-Approx}(H,\frac{SX}{H},T,r,D,\sigma)$$

Where the right hand term is the equivalent vanilla American option price calculated by using an approximation. Similar values can be given to other types of American barriers.

In an attempt to reduce the complexity of pricing path dependent options like barriers, many have used variance reduction techniques to reduce computation time of barriers and arise at faster convergence. A notable look into variance reduction, in particular, the use of importance sampling was undertaken by Glasserman & Shaum (1999) on "out" type barriers.

Reiner & Rubinstein (1992) suggest the use of two distinct volatilities to price barrier options, similar to that of pricing a two-asset barrier, which we look at in a separate section. The use of two distinct volatilities is an attempt to reduce the bias when only one volatility is considered and is often used in trading.

Please see respective sections for pricing of double barriers, parisian barriers, partial time barriers and other exotic barrier types.

9) Other Known Names / Variants

Double Barriers
Knock-In Options
Knock-Out Options
Look Barriers
Parisian Options
Partial Time Barriers
Parasian Options
Rainbow /Two-Asset Barriers
Reverse Barrier Options
Soft / Fluffy Barriers
Window Barriers

References:

Barone-Adesi, G., & R.E. Whaley "Efficient Analytic Approximation of American Option Values", Journal of Finance, 42, 301-320. June 1987

Beaglehole, D.R., Dybvig, P.H., Zhou, G., "Going to Extremes: Correcting Simulation Bias in Exotic Option Valuation", Assocation for Investment Management & Research Paper (1997)

Black, F. & Scholes, M., "The Pricing of Options & Corporate Liabilities", The Journal of Political Economy (May '73)

Broadie, M., Glasserman, P., Kou, S.G., "A Continuity Correction for Discrete Barrier Options", Journal of Mathematical Finance, (Oct '97)

Bjerksund P. & Stensland G., "Closed-Form Approximation of American Options", Scandinavian Journal of Management, vol.9, 1993, pp.87-99

Boyle, P., & Lau, S.H., "Bumping Against the Barrier with the Binomial Method", Journal of Derivatives, 1, 6-14 (1994)

Boyle, P., "Option Valuation Using a Three-Jump Process", International Options Journal 3, pages 7-12 (1986)

Cox, J., Ross, S. and Rubinstein M. "Option Pricing: A Simplified Approach." Journal of Financial Economics, 7. (Sep '79)

Derman, E., Kani, I., Ergener, D., & Bardhan, I., "Enhanced Numerical Methods for Options With Barriers", Goldman Sachs Working Paper (May '95)

Derman, E. & Kani. I., "Riding on a Smile." Risk 7, 2 pp. 32-39 (Feb '94)

Figlewski S., & Gao, B., "The Adaptive Mesh Model: A New Approach to Efficient Option Pricing", Journal of Financial Economics, 53, pp. 313-51, 1999

Figlewski S., Gao, B., Ahn., D.H., "Pricing Discrete Barrier Options with an Adaptive Mesh Model", Working Paper, 1999

Glasserman, P., & Staum, J., "Condition on One-Step Survival for Barrier Option Simulations", Operations Research (Jul '99)

Haug, E., "Complete Guide to Option Pricing Formulas", McGraw Hill (1998)

Hull, J., "Options, Futures & Other Derivatives", 5th Edition 2002 - Chapter 19-20

Kou, S., "On Pricing of Discrete Barrier Options", University of Columbia Working Paper & Statistica Sinica, 2001

Leisen, D., "Valuation of Barrier Options in a Black-Scholes Setup with Jump Risk" European Finance Review 3, 3, 319-342 (1999)

Levitan, S., "Lattice Methods for Barrier Options", University of Witwatersrand Honours Project (Dec '01)

Levitan, S., Mitchell, K., Taylor, D.R., "Discrete Closed Form Solutions for Barrier Options" University of Witwatersrand Honours Project (2003)

Merton, R., "Theory of Rational Option Pricing", Bell Journal of Economics & Management (June '73)

Metwally, S., & Atiya, A., "Using Brownian Bridge for Fast Simulation of Jump-Diffusion Processes and Barrier Options" Journal of Derivatives (Fall '02)

Reiner, E. & Rubinstein, M., "Breaking Down the Barriers", Risk 4, 8, pp. 28-35 (Sep 1991)

Reiner, E., & Rubinstein, M., "Exotic Options", Working Paper 1992

Rubinstein, M., "On the Relation of Binomial and Trinomial Option Pricing Models" Journal of Derivatives (Winter 2000)