Completion of a Lévy Market by Power-Jump Assets[*]

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Abstract

We work under a geometric Lévy market model: the stock price process is modelled by a SDE driven by a general Lévy process (taking into account jumps). Except for the geometric Brownian model and the geometric Poissonian model, the above described general geometric Lévy market models are incomplete models and there are many equivalent martingale measures.

In this paper we suggest to enlarge the market by a series of very special assets (power-jump assets) related to the power-jump processes of the underlying Lévy process suitable compensated. By doing this we show that the market can be completed. The very particular choice of the compensators needed to make these processes tradable is very delicate. For some special cases, we will derive criteria to ensure their introduction in the market will not lead to arbitrage opportunities. The question in general is related to the moment problem.

Keywords: LÉVY PROCESSES, MARKET MODELS, MARTINGALES, SDE, VARIATION, ARBITRAGE, COMPLETE MARKETS, MOMENT PROBLEM, ORTHOGONAL POLYNOMIALS

1 Introduction

We will work under a so-called (geometric) Lévy market model. Under this model the stock price process $S = \{S_t, t \geq 0\}$ is modelled by a Stochastic Differential Equation (SDE) driven by a general Lévy process $Z = \{Z_t, t \geq 0\}$

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(satisfying some conditions):

$$\frac{\mathrm{d}S_t}{S_{t-}} = b\mathrm{d}t + \mathrm{d}Z_t, \qquad S_0 > 0. \tag{1}$$

The classical Black-Scholes model [6] is taking for the Lévy process Z a Brownian motion. We will allow more general Lévy processes (taking into account jumps). The classical Black-Scholes model is a so-called *complete* model, in that all contingent claims can be duplicated by a portfolio consisting of investments in the stock and in a risk-free bond: the risk of any claim can be completely hedged against. In such a complete model there exists a unique equivalent martingale measure and the unique price of a contingent claim is just the discounted expectation of the payoff at maturity.

Except for the geometric Brownian model and the geometric Poissonian model, for the above described general geometric Lévy market models, there are many equivalent martingale measures and such markets are incomplete: contingent claims cannot in general be hedged by a portfolio.

In this paper we suggest to enlarge the market by a series of very special assets related to the power-jump processes of the underlying Lévy process. These processes were introduced in [21] and are related to the (realized) variation processes as studied in [3] and [4]. However the very particular choice of the compensators and their orthonormalized version is very delicate. For some special cases, we will derive criteria to ensure their introduction in the market will not lead to arbitrage opportunities. A Martingale Representation Theorem in terms of these (orthonomalized) power-jump processes leads us to market completeness.

For pure jump processes, the power-jump process of order two is just the variation process of degree two, i.e. the quadratic variation process (see e.g. [2], [3] and [4]), and is related with the so-called realized variance. Contracts on realized variance have found their way into OTC markets and are now traded regularly. Higher order power-jump processes have a similar relationship with which one could call realized skewness and realized kurtosis processes. Contracts on these objects however are not common. Carr, Geman, Madan and Yor [10] study contracts on the quadratic variation processes in a model driven by a so-called Sato process. Other references about Realized Variance products are [9], [12], [14] and [8].

First we recall some basic results on Lévy processes. Next, we introduce the power-jump processes, their compensated versions, the orthonormailzed processes and the assets by which we will enlarge the market. In a first part we take as given an equivalent martingale measure, which we belief is the right one. We then introduce into the market a series of assets based on the above mentioned processes, in a way that the market will remain arbitrage-free. We then show that the market is also complete and given explicit hedging portfolios for claims which payoff function only depends on the stock prices at maturity. In a second part we reverse the question and assume the market is already enlarged by these so-called (orthonormalized) power-jump assets. Here we turn to the question whether the market is arbitrage-free. We look whether there exists an equivalent martingale measure making all the discounted traded assets martingales. We thus study the existence of an equivalent martingale measure making not only the discounted stock price a martingale, but also all discounted newly introduced assets. This question is related to the moment problem. We make the computation explicit in a few popular examples.

2 The Geometric Lévy Model

Before describing the model we first recall some basic results on Lévy processes. Basic reference texts on Lévy processes are [5] and [25]. For applications in finance see [26]. We closely follow the exposure given in [11].

2.1 Lévy Processes

Suppose $\phi(z)$ is the characteristic function of a distribution. If for every positive integer n, $\phi(z)$ is also the nth power of a characteristic function, we say that the distribution is infinitely divisible. One can define for every such an infinitely divisible distribution a stochastic process, $Z = \{Z_t, t \geq 0\}$, called Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over $[s, s+t]$, $s, t \geq 0$, i.e. $Z_{t+s} - Z_s$, has $(\phi(z))^t$ as characteristic function. It is well known that Lévy processes are semimartingales.

Such a real-valued stochastic process Z, defined in a complete probability space $(\Omega, \mathcal{F}, P)$ has a càdlàg modification [23, Theorem 30, page 21] and we will always assume that we are using this càdlàg version. If we let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$, where $\mathcal{G}_t = \sigma\{Z_s, 0 \leq s \leq t\}$ is the natural filtration of Z, and $\mathcal{N}$ are the P-null sets of $\mathcal{F}$, then $\{\mathcal{F}_t, t \geq 0\}$ is a right continuous family of σ-fields [23, Theorem 31, page 22].

The function $\psi(z) = \log \phi(z) = \log E[\exp(\mathrm{i}zZ_1)]$ is called the *characteristic exponent* and it satisfies the following *Lévy-Khintchine formula* [5]:

$$\psi(z) = \mathrm{i}\alpha z - \frac{c^2}{2}z^2 + \int_{-\infty}^{+\infty} (\exp(\mathrm{i}zx) - 1 - \mathrm{i}zx1_{\{|x|<1\}})\nu(\mathrm{d}x),$$

where $\alpha \in \mathbb{R}$, $c \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with $\int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$. We say that our infinitely divisible distribution has a triplet of Lévy characteristics $[\alpha, c^2, \nu(\mathrm{d}x)]$. The measure $\nu(\mathrm{d}x)$ is called the *Lévy measure* of Z, $\nu(\mathrm{d}x)$ dictates how the jumps occur. Jumps of sizes in the set A occur according to a Poisson process with parameter $\int_A \nu(\mathrm{d}x)$. If $c^2 = 0$ and $\int_{-1}^{+1} |x|\nu(\mathrm{d}x) < \infty$ it follows from standard Lévy process theory (see [5] and [25]), that the process is of finite variation.

From the Lévy-Khintchine formula, one can deduce that Z must be a linear combination of a standard Brownian motion $W = \{W_t, t \geq 0\}$ and a pure jump process $X = \{X_t, t \geq 0\}$:

$$Z_t = cW_t + X_t,$$

and where W is independent of X. Moreover

$$X_t = \int_{\{|x|<1\}} x(Q((0,t], \mathrm{d}x) - t\nu(\mathrm{d}x)) + \int_{\{|x|\geq 1\}} xQ((0,t], \mathrm{d}x) + \alpha t,$$

where $Q(\mathrm{d}t, \mathrm{d}x)$ is a Poisson random measure on $(0, +\infty) \times \mathbb{R} \backslash \{0\}$ with intensity $\mathrm{d}t \times \nu$, where ν is the Lévy measure of Z and $\mathrm{d}t$ denotes the Lebesgue measure.

For the purpose of our model we require the process Z to satisfy certain conditions. We will suppose that the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\nu(dx) < \infty.$$

This implies that

$$\int_{-\infty}^{+\infty} |x|^i \nu(\mathrm{d}x) < \infty, \qquad i \geq 2, \tag{2}$$

and that the characteristic function $E[\exp(\mathrm{i}uX_t)]$ is analytic in a neighborhood of 0 and

$$E[\exp(-hZ_1)] < \infty \text{ for all } h \in (-h_1, h_2), \tag{3}$$

where $0 < h_1, h_2 \leq \infty$. So all moments of Z_t (and X_t) exist. Note that

$$\alpha = E[X_1] - \int_{|x|\geq 1} x\nu(\mathrm{d}x).$$

With these assumptions one can see that the Doob decomposition of X, in terms of a martingale part and a predictable process of finite variation, is given by

$$X_t = L_t + at,$$

where $L = \{L_t, t \geq 0\}$ is a martingale and $E[X_1] = a$. We refer to the process $t \mapsto at$ as the predictable part or the compensator of X.

If we write $M(\mathrm{d}t, \mathrm{d}x) = Q(\mathrm{d}t, \mathrm{d}x) - \mathrm{d}t\nu(\mathrm{d}x)$ the martingale part of X can be written in terms of the compensated Poisson random measure $M(\mathrm{d}t, \mathrm{d}x)$ as

$$L_t = \int_{-\infty}^{+\infty} xM((0,t], \mathrm{d}x).$$

2.2 Power-Jump Processes

The following transformations of $Z = \{Z_t, t \geq 0\}$ will play an important role in our analysis. We set

$$Z_t^{(i)} = \sum_{0<s\leq t} (\Delta Z_s)^i, \qquad i \geq 2,$$

where $\Delta Z_s = Z_s - Z_{s-}$, and for convenience we put $Z_t^{(1)} = Z_t$. Note that not necessarily $Z_t = \sum_{0<s\leq t} \Delta Z_s$ holds; it is only true in the bounded variation case

(with c necessarily equal to zero). Note also that, $X_t^{(i)} = Z_t^{(i)}$, $i \geq 2$ and clearly $[X, X]_t = X_t^{(2)}$. The process $X^{(i)} = \{X_t^{(i)}, t \geq 0\}$, $i = 2, 3, \ldots$, is again a Lévy process and is called the ith-power-jump processes (or the power-jump process of order i). They jump at the same points as the original Lévy process, but the jumps sizes are the ith power of the jump size of the original Lévy process.

We have $E[X_t] = E[X_t^{(1)}] = ta = tm_1 < \infty$ and (see [23, page 29])

$$E\left[X_t^{(i)}\right] = E\left[\sum_{0<s\leq t} (\Delta X_s)^i\right] = t\int_{-\infty}^{\infty} x^i \nu(\mathrm{d}x) = m_i t < \infty, \qquad i \geq 2. \tag{4}$$

We denote by

$$Y_t^{(i)} = Z_t^{(i)} - E\left[Z_t^{(i)}\right] = Z_t^{(i)} - m_i t, \qquad i = 1, 2, 3, \ldots.$$

the compensated ith-power-jump processes. $Y^{(i)} = \{Y_t^{(i)}, t \geq 0\}$ is a normal martingale, and was called the *Teugels martingale of order i*.

In [21] a procedure is described to orthonormalize the sequence of martingales $\{Y^{(i)}, i = 1, 2, \ldots\}$. By taking a suitable linear combination of the $Y^{(i)}$'s, one obtains a set of pairwise strongly orthonormal martingales $\{T^{(i)}, i \geq 1\}$. Each $T^{(i)}$ is a linear combinations of the $Y^{(j)}, j = 1, 2, \ldots, i$:

$$T^{(i)} = c_{i,i} Y^{(i)} + c_{i,i-1} Y^{(i-1)} + \ldots + c_{i,1} Y^{(1)}, \qquad i \geq 1.$$

The constants $c_{i,j}$ can be calculate as described in [21]: they correspond to the coefficients of the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(\mathrm{d}x) = x^2\nu(\mathrm{d}x) + c^2\delta_0(\mathrm{d}x)$. The resulting processes $T^{(i)} = \{T_t^{(i)}, t \geq 0\}$ are called the *orthonormalized ith-power-jump processes.*

Remark 1 *In the case of a Poisson process N_t with parameter $\lambda > 0$, all Teugels martingales are equal to $N_t - \lambda t$. Hence, $T_t^{(1)} = \frac{N_t - \lambda t}{\sqrt{\lambda}}$, and $T_t^{(i)} = 0$ for all $i \geq 2$.*

Remark 2 *In the case of a Brownian motion, all (orthonormized) ith-power-jump processes, $i = 2, 3, \ldots$, are equal to zero.*

2.3 The Geometric Lévy Model

Using Itô's formula (see [11] or [23]) for càdlàg semimartingales one can show that Equation (1) has an explicit solution

$$S_t = S_0 \exp\left(cW_t + L_t + \left(a + b - \frac{c^2}{2}\right)t\right) \prod_{0<s\leq t} (1 + \Delta L_s)\exp(-\Delta L_s).$$

In order to ensure that $S_t \geq 0$ for all $t \geq 0$ almost surely, we need $\Delta L_t \geq -1$ for all t. We thus need that the Lévy measure ν is supported on a subset of $[-1, +\infty]$.

The riskless rate of interest we assume to be a constant r. The value of the riskfree bond or bank account at time t is then given by $B_t = \exp(rt)$.

3 Enlarging the Lévy Market Model

Suppose we have an equivalent martingale measure Q under which Z remains a Lévy process (see comment below). Under this measure the discounted stock price process is a martingale and the process $\tilde{Z} = \{Z_t + (b-r)t, t \geq 0\}$ will be a Lévy process (with under Q a Lévy measure $\tilde{\nu}$ say); moreover the process $\tilde{Z}$ is a martingale. Obviously $\Delta \tilde{Z}_t = \Delta Z_t$ and $\tilde{Z}_t^{(i)} = Z_t^{(i)}$, $i \geq 2$. Let us consider (based on $\tilde{Z}$) the ith-power-jump processes $Y^{(i)} = \{Y_t^{(i)}, t \geq 0\}$ and their orthonormalized version $T^{(i)} = \{T_t^{(i)}, t \geq 0\}$ under the measure Q, i.e. the compensators are $m_i t = t E_Q[\tilde{Z}_1^{(i)}]$ and the orthonormalization procedure is performed under Q. Note that for $i \geq 2$, $m_i = \int_{-\infty}^{+\infty} x^i \tilde{\nu}(\mathrm{d}x)$, and we will require $\tilde{\nu}$ to fulfill (2).

Next, we will enlarge the Lévy Market with a series of additional assets based on the above mentioned processes. We consider two cases which are related but have very subtle differences. On one hand we consider assets based on the orthonormal processes $T^{(i)}$, which allow us to write a series decomposition for any square integrable payoff. Secondly, we deal with the non-orthogonalized processes $Y^{(i)}$, in this case, the series development is of Taylor-type and it requires some regularity conditions on the payoff function.

First we enlarge the market with what we will call the *orthonormalized ith-power-jump assets*. More precisely we will allow trade in assets with price process $\bar{H}^{(i)} = \{\bar{H}_t^{(i)}, t \geq 0\}$, where

$$\bar{H}_t^{(i)} = \exp(rt) T_t^{(i)}, \qquad i = 2, 3, \ldots .$$

Next, we consider an enlargement based on assets with price process $H^{(i)} = \{H_t^{(i)}, t \geq 0\}$, where

$$H_t^{(i)} = \exp(rt) Y_t^{(i)}, \qquad i = 2, 3, \ldots .$$

We will call these the *ith-power-jump assets*. This enlargement is in some sense more straightforward.

Trade in the power-jump assets can be motivated as follows. Consider the 2th-power-jump asset. This object in some sense measures the volatility of the stock, since it accounts for the square of the jumps. If one believes that in the future there will be a more volatile environment than the current market's anticipation, trading the 2th-power-jump asset can be of interest. Also if one would like to cover against periods of high (or low) volatility, they can be useful: Buying 2th-power-jump assets can cover the possible losses due to such unfavorable periods. The same can be said for the higher order variation assets. Typically the 3th-power-jump assets is measuring a kind of asymmetry (cfr. skewness) and the 4th-power-jump process is measuring extremal movements (cfr. kurtosis). Trade in these assets can be of use if one likes to bet on the realized skewness or realized kurtosis of the stock: one believes that the market is not counting in asymmetry and possible extremal moves rightly. On the other hand, an insurance against a crash can be easily build from the 4th-power-jump (or ith-power jump, $i \geq 4$) assets.

Note that clearly, the discounted versions of the $H^{(i)}$ (and $\bar{H}^{(i)}$) are the (orthonormailzed) power-jump processes, and hence martingales:

$$E_Q[\exp(-rt)\bar{H}_t^{(i)}|\mathcal{F}_s] = E_Q[T_t^{(i)}|\mathcal{F}_s] = T_s^{(i)}, \quad 0 \leq s \leq t,$$

and

$$E_Q[\exp(-rt)H_t^{(i)}|\mathcal{F}_s] = E_Q[Y_t^{(i)}|\mathcal{F}_s] = Y_s^{(i)}, \quad 0 \leq s \leq t.$$

Hence the market allowing trade in the bond, the stock and the (orthonormalized) power-jump assets remains arbitrage-free.

3.1 Enlarging the Lévy Market Model with Orthonormalized Power-Jump Assets

3.2 Market Completeness

Next, we will show using the Martingale Representation Property (MRP) for Lévy processes derived in [21] and further exploited in [22] that our enlarged market is complete. In fact this MRP in terms of the orthonormalized power-jump processes lies at the heart of this exposition. The idea of enlarging the market by these objects comes directly from their appearance in the MRP.

So, suppose that we have selected an equivalent martingale measure Q under which all discounted assets price processes are martingales. Then we claim that the market is complete in the sense that for every square integral contingent claim X (i.e. a non-negative square-integrable $\mathcal{F}_T$-measurable random variable), we can set up a self-financing tame portfolio replicating the claim perfectly. The portfolio can consists of a number of bonds, stocks and orthonormalized ith-power-jump assets, $i = 2, 3, \ldots$.

We will make use of the MRP obtained in [21] which says that a square-integrable martingale M_t can be represented as follows:

$$M_t = M_0 + \int_0^t h_s \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} \mathrm{d}T_s^{(i)},$$

where h_s and $h_s^{(i)}$, $i = 2, 3, \ldots$ are predictable processes, such that

$$E[\int_0^t |h_s|^2 \mathrm{d}s] < \infty$$

and

$$E[\int_0^t \sum_{i=2}^{\infty} |h_s^{(i)}|^2 \mathrm{d}s] < \infty. \tag{5}$$

Theorem 3 *The Lévy market model enlarged with the orthonormalized ith-power-jump assets is complete, in the sense that any square-integrable contingent claim X can be replicated.*

Proof: Consider a square-integrable contingent claim X with maturity T. Let

$$M_t = E_Q[\exp(-rT)X|\mathcal{F}_t].$$

We apply to this martingale the MRP given above:

$$M_t = M_0 + \int_0^t h_s \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} \mathrm{d}T_s^{(i)}.$$

Next, we look for a self-financing tame strategy

$$\phi = \{\phi_t = (\alpha_t, \beta_t, \beta_t^{(2)}, \beta_t^{(3)}, \ldots), t \geq 0\}$$

to replicate X. Here α_t corresponds with the number of bonds at time t; β_t is the number of stocks at that time and $\beta_t^{(i)}$ is the number of assets $\bar{H}^{(i)}$, $i = 2, 3, \ldots$, one needs to hold at time t. We claim that in terms of the ingredients of the MRP

$$\begin{aligned}
\alpha_t &= M_{t-} - \beta_t S_{t-} B_t^{-1} - \sum_{i=2}^{\infty} \beta_t^{(i)} \bar{H}_{t-}^{(i)} B_t^{-1} \\
\beta_t &= h_t B_t S_{t-}^{-1} \\
\beta_t^{(i)} &= h_t^{(i)}, \qquad i = 2, 3, \ldots.
\end{aligned}$$

Notice that condition (5) and the fact that the random variables $T_t^{(i)}$ are orthonormal in $L^2(\Omega)$, ensure that the series appearing in the definition of α_t is convergent in mean square, for almost all t.

We have

$$\begin{aligned}
\Delta M_t &= h_t \Delta Z_t + \sum_{i=2}^{\infty} h_t^{(i)} \Delta T_t^{(i)} \\
\Delta S_t &= S_{t-} \Delta Z_t \\
B_t \Delta T_t^{(i)} &= \Delta \bar{H}_t^{(i)}, \quad i = 2, 3, \ldots.
\end{aligned}$$

Hence, one sees that the value V_t of such a portfolio at time t is given by

$$\begin{aligned}
V_t &= \alpha_t B_t + \beta_t S_t + \sum_{i=2}^{\infty} \beta_t^{(i)} \bar{H}_t^{(i)} \\
&= B_t M_{t-} + \beta_t \Delta S_t + \sum_{i=2}^{\infty} \beta_t^{(i)} \Delta \bar{H}_t^{(i)} \\
&= B_t M_t
\end{aligned}$$

which is the price of the claim at time t. So the portfolio is replicating the claim. Moreover, clearly $V_t \geq 0$ so the strategy ϕ is also tame.

Next, we show that ϕ is self-financing. Let

$$G_u = \int_0^u \alpha_t \mathrm{d}B_t + \int_0^u \beta_t \mathrm{d}S_t + \sum_{i=2}^{\infty} \int_0^u \beta_t^{(i)} \mathrm{d}\bar{H}_t^{(i)}$$

denote the gain process, i.e. the gains or losses obtained up to time u by following ϕ. We will show that

$$G_u + M_0 = M_u B_u,$$

which implies the portfolio is self-financing.

We have

$$\begin{aligned} G_u &= \int_0^u M_{t-} \mathrm{d}B_t - \int_0^u h_t \mathrm{d}B_t - \sum_{i=2}^{\infty} \int_0^u h_t^{(i)} \bar{H}_{t-}^{(i)} B_t^{-1} \mathrm{d}B_t \\ &\quad + \int_0^u h_t B_t S_{t-}^{-1} \mathrm{d}S_t + \sum_{i=2}^{\infty} \int_0^u h_t^{(i)} \mathrm{d}\bar{H}_t^{(i)}. \end{aligned}$$

Now,

$$\begin{aligned} &\int_0^u M_t \mathrm{d}B_t \\ &= \int_0^u \left(M_0 + \int_0^{t-} h_s \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^{t-} h_s^{(i)} \mathrm{d}T_s^{(i)} \right) \mathrm{d}B_t \\ &= M_0(B_u - B_0) + \int_0^u \int_0^{t-} h_s \mathrm{d}\tilde{Z}_s \mathrm{d}B_t + \sum_{i=2}^{\infty} \int_0^u \int_0^{t-} h_s^{(i)} \mathrm{d}T_s^{(i)} \mathrm{d}B_t \\ &= M_0(B_u - B_0) + \int_0^u \int_{s+}^u h_s \mathrm{d}B_t \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^u \int_{s+}^u h_s^{(i)} \mathrm{d}B_t \mathrm{d}T_s^{(i)} \\ &= M_0(B_u - B_0) + \int_0^u h_s (B_u - B_s) \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^u h_s^{(i)} (B_u - B_s) \mathrm{d}T_s^{(i)} \\ &= M_0(B_u - B_0) + B_u \int_0^u h_s \mathrm{d}\tilde{Z}_s - \int_0^u h_s B_s \mathrm{d}\tilde{Z}_s \\ &\quad + B_u \sum_{i=2}^{\infty} \int_0^u h_s^{(i)} \mathrm{d}T_s^{(i)} - \sum_{i=2}^{\infty} \int_0^u h_s^{(i)} B_s \mathrm{d}T_s^{(i)} \\ &= M_0(B_u - B_0) + B_u(M_u - M_0) - \int_0^u h_s B_s \mathrm{d}\tilde{Z}_s - \sum_{i=2}^{\infty} \int_0^u h_s^{(i)} B_s \mathrm{d}T_s^{(i)} \end{aligned}$$

Hence,

$$\begin{aligned} G_u &= M_u B_u - M_0 \\ &\quad - \int_0^u h_t \mathrm{d}B_t - \sum_{i=2} \int_0^u h_t^{(i)} \bar{H}_{t-}^{(i)} B_t^{-1} \mathrm{d}B_t \end{aligned}$$

$$
\begin{aligned}
& -\int_0^u h_t B_t \mathrm{d}\tilde{Z}_t + \int_0^u h_t B_t S_{t-}^{-1} \mathrm{d}S_t \\
& +\sum_{i=2}^{\infty} \int_0^u h_t^{(i)} \mathrm{d}\bar{H}_t^{(i)} - \sum_{i=2}^{\infty} \int_0^u h_t^{(i)} B_t \mathrm{d}T_t^{(i)} \\
= \;& M_u B_u - M_0 - \int_0^u h_t \mathrm{d}B_t - \sum_{i=2} \int_0^u h_t^{(i)} \bar{H}_{t-}^{(i)} B_t^{-1} \mathrm{d}B_t \\
& -\int_0^u h_t B_t \mathrm{d}\tilde{Z}_t + \int_0^u h_t B_t S_{t-}^{-1} \mathrm{d}S_t + \sum_{i=2}^{\infty} \int_0^u h_t^{(i)} T_{t-}^{(i)} \mathrm{d}B_t \\
= \;& M_u B_u - M_0 - \int_0^u h_t \mathrm{d}B_t \\
& -\int_0^u h_t B_t S_{t-}^{-1} \mathrm{d}S_t + \int_0^u h_t \mathrm{d}B_t + \int_0^u h_t B_t S_{t-}^{-1} \mathrm{d}S_t \\
= \;& M_u B_u - M_0
\end{aligned}
$$

This completes the proof. •

3.3 Enlarging the Lévy Market Model with Power-Jump Assets

Next, we consider a market enlarged with the power-jump assets. Note that their discounted price processes are martingales, but these martingales are no longer strongly orthogonal. This fact implies that a square-integrable martingale can in general not always be represented as in the MRP but now with as integrands the processes $Y^{(i)}$.

However, in some particular examples one can represent a square-integrable martingales M in terms of the $Y^{(i)}$:

$$
M_t = M_0 + \int_0^t h_s \mathrm{d}\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} \mathrm{d}Y_s^{(i)},
$$

where h_s and $h_s^{(i)}$, $i = 2, 3, \ldots$ are predictable processes satisfying $E[\int_0^t h_s \mathrm{d}s]^{1/2} < \infty$ and

$$
\sum_{i=2}^{\infty} \left(E\left[\int_0^t h_s^{(i)} \mathrm{d}s \right] \right)^{1/2} \langle Y^{(i)} \rangle t^{-1} < \infty.
$$

An example of this kind of representation is the case

$$
M_t = E_Q(F(t, S_T) | \mathcal{F}_t),
$$

where the function $F(t, x) \in C^{1,\infty}$ satisfies condition (8) below.

Moreover, as shown in [18], we have that if the Lévy measure is concentrated in a finite number of points, j say, then the orthonormal martingales $H^{(k)}$ vanish for all $j \geq k+1$ if there is a no continuous component and for all $j \geq k+2$ if there is a continuous component. In such cases the representation using orthonormal martingales is finite, and as a consequence, we also obtain a finite representation using the (non-orthogonal) martingales $Y^{(i)}$. The example work out in Section (4.2.2) is of this type.

3.4 Hedging Portfolios

Next, we show how to calculate explicitly the hedging portfolio of a contingent claim of which the payoff is only a function of the value at maturity of the stock price, i.e. $X = F(T, S_T)$. We work under the framework of the market enlarged with the ith-power-jump assets. A similar analyze can be made under the market enlarged with the orthonormalized power-jump assets.

We start with noting that the value of the contingent claim at time t is given by:

$$F(t, S_t) = \exp(-r(T-t))E_Q[X|\mathcal{F}_t];$$

we call $F(t, x)$ the price function of X.

Denote by D_1 the differential operation with respect to the first variable, i.e. the time variable, and by D_2 the differential operator with respect to the space variable (the second variable : the stock price). Finally, denote by $\mathcal{D}$ the following integral operator:

$$\mathcal{D}F(t,x) = \int_{-\infty}^{+\infty} \left(F(t, x(1+y)) - F(t,x) - xyD_2F(t,x)\right)\tilde{\nu}(\mathrm{d}y).$$

It is well known [11] that in analogy with the Black-Scholes partial differential equation, F will satisfy under a Lévy market setting a Partial Differential Integral Equation (PDIE). More precisely, the price function (at time t) $F(t, x)$ satisfies:

$$D_1F(t,x) + rxD_2F(t,x) + \frac{1}{2}c^2x^2D_2^2F(t,x) + \mathcal{D}F(t,x) = rF(t,x). \qquad (6)$$

We will need the following lemma:

Lemma 4 *Consider a real function $h(s, x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ which is infinitely differentiable in the y variable. Set*

$$a_i(s,x) = \frac{1}{i!}\frac{\partial^i h}{\partial y^i}(s,x,0),$$

and assume that

$$\sup_{x<K, s\leq s_0} \sum_{i=2}^{\infty} |a_i(s,x)|R^i < \infty, \qquad (7)$$

for all $K, R > 0$, $s_0 > 0$. Then we have

$$\sum_{t<s\leq T} h(s, S_{s-}, \Delta X_s) = \sum_{i=2}^{\infty}\int_t^T \frac{1}{i!}\frac{\partial^i}{\partial y^i}h(s, S_{s-}, 0)\,dY_s^{(i)} + \int_t^T\int_{-\infty}^{\infty} h(s, S_{s-}, y)\tilde{\nu}(dy)ds$$

Proof: The function $h(s,x,y)$ can be expanded as

$$h(s,x,y)=\sum_{i=2}^{\infty}a_i(s,x)y^i.$$

Then we have

$$\begin{aligned}\sum_{t<s\le T}h(s,S_{s-},\Delta X_s) &= \sum_{t<s\le T}\sum_{i=2}^{\infty}a_i(s,S_{s-})(\Delta X_s)^i\\ &= \sum_{i=2}^{\infty}\int_t^T a_i(s,S_{s-})\mathrm{d}X_s^{(i)}\\ &= \sum_{i=2}^{\infty}\int_t^T a_i(s,S_{s-})\mathrm{d}Y_s^{(i)}\\ &\quad+\sum_{i=2}^{\infty}\int_t^T a_i(s,S_{s-})m_i\mathrm{d}s,\end{aligned}$$

where the sums converge for every $\omega\in\Omega$ because of (7).

Using that $m_i=\int_{-\infty}^{\infty}y^i\tilde{\nu}(\mathrm{d}y)$, we have

$$\begin{aligned}\sum_{t<s\le T}h(s,S_{s-},\Delta X_s) &= \sum_{i=2}^{\infty}\int_t^T a_i(s,S_{s-})\mathrm{d}Y_s^{(i)}\\ &\quad+\int_t^T\int_{-\infty}^{\infty}\sum_{i=2}^{\infty}a_i(s,S_{s-})y^i\tilde{\nu}(\mathrm{d}y)\mathrm{d}s\\ &= \sum_{i=2}^{\infty}\int_t^T a_i(s,S_{s-})\mathrm{d}Y_s^{(i)}\\ &\quad+\int_t^T\int_{-\infty}^{\infty}h(s,S_{s-},y)\tilde{\nu}(\mathrm{d}y)\mathrm{d}s\end{aligned}$$

This proves the lemma. •

Now, we calculate the hedging portfolio for the contingent claim X.

Theorem 5 *The (tame and self-financing) replicating strategy of a contingent claim X, with a payoff only depending on the stock price value at maturity and a price function $F(t,x)\in C^{1,\infty}$ which satisfies*

$$\sup_{x<K,t\le t_0}\sum_{n=2}^{\infty}|D_2^nF(t,x)|R^n<\infty, \tag{8}$$

for all $K,R>0$, $t_0>0$, is given at time t by:

$$\textit{number of bonds}=\alpha_t \;=\; B_t^{-1}\left(F(t,S_{t-})-S_{t-}D_2F(t,S_{t-})\right))$$

$$
\begin{aligned}
& & -B_t^{-1}\sum_{i=2}^{\infty}\frac{S_{t-}^i D_2^i F(t,S_{t-})}{i!B_t}H_{t-}^{(i)} \\
\textit{number of stocks} = \beta_t &=& D_2F(t,S_{t-}) \\
\textit{number of ith-power-jump assets} = \beta_t^{(i)} &=& \frac{S_{t-}^i D_2^i F(t,S_{t-})}{i!B_t}, \qquad i=2,3,\ldots.
\end{aligned}
$$

Proof: First, apply Itô's lemma to $F(T,S_T)$. This gives

$$
\begin{aligned}
F(T,S_T)-F(t,S_t) &= \int_t^T D_1F(s,S_{s-})\mathrm{d}s+\frac{c^2}{2}\int_t^T S_{s-}^2D_2^2F(s,S_{s-})\mathrm{d}s \\
&\quad +\int_t^T D_2F(s,S_{s-})\mathrm{d}S_s \\
&\quad +\sum_{t<s\le T}F(s,S_s)-F(s,S_{s-})-D_2F(s,S_{s-})\Delta S_s
\end{aligned}
$$

Noting that $\Delta S_s = S_{s-}\Delta X_s$ this becomes

$$
\begin{aligned}
&F(T,S_T)-F(t,S_t) \\
&\quad = \int_t^T D_1F(s,S_{s-})\mathrm{d}s+\frac{c^2}{2}\int_t^T S_{s-}^2D_2^2F(s,S_{s-})\mathrm{d}s \\
&\qquad +\int_t^T D_2F(s,S_{s-})\mathrm{d}S_s \\
&\qquad +\sum_{t<s\le T}F(s,S_{s-}(1+\Delta X_s))-F(s,S_{s-})-D_2F(s,S_{s-})\Delta X_sS_{s-}
\end{aligned}
$$

Let $h(t,x,y)=F(t,x(1+y))-F(t,x)-xyD_2F(t,x)$. Then

$$
\begin{aligned}
h(t,x,0) &= 0 \\
\frac{\partial}{\partial y}h(t,x,0) &= 0 \\
\frac{\partial^n}{\partial y^n}h(t,x,0) &= x^nD_2^nF(t,x), \quad n=2,3,\ldots
\end{aligned}
$$

Since $F(t,x)\in C^{1,\infty}$ satisfies condition (8), then h is satisfying the conditions in Lemma 4. Applying Lemma 4 gives

$$
\begin{aligned}
&F(T,S_T)-F(t,S_t) \\
&\quad = \int_t^T D_1F(s,S_{s-})\mathrm{d}s+\frac{c^2}{2}\int_t^T S_{s-}^2D_2^2F(s,S_{s-})\mathrm{d}s \\
&\qquad +\int_t^T D_2F(s,S_{s-})\mathrm{d}S_s+\sum_{i=2}^{\infty}\int_t^T\frac{S_{s-}^iD_2^iF(s,S_{s-})}{i!}\mathrm{d}Y_s^{(i)} \\
&\qquad +\int_t^T\int_{-\infty}^{\infty}F(t,x(1+y))-F(t,x)-xyD_2F(t,x)\tilde{\nu}(\mathrm{d}y)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \int_t^T \frac{D_1F(s,S_{s-}) + S_{s-}^2\frac{c^2}{2}D_2^2F(s,S_{s-}) + \mathcal{D}F(t,x)}{rB_s}\mathrm{d}B_s \\
& + \int_t^T D_2F(s,S_{s-})\mathrm{d}S_s + \sum_{i=2}^{\infty}\int_t^T \frac{S_{s-}^iD_2^iF(s,S_{s-})}{i!B_s}\mathrm{d}H_s^{(i)} \\
& - \sum_{i=2}^{\infty}\int_t^T \frac{S_{s-}^iD_2^iF(s,S_{s-})}{i!B_s}Y_{s-}^{(i)}\mathrm{d}B_s
\end{aligned}
$$

Finally, using the PDIE equation (6) for the price, we obtain

$$
\begin{aligned}
& F(T,S_T) - F(t,S_t) \\
& \quad = \int_t^T \frac{F(s,S_{s-}) - S_{s-}D_2F(s,S_{s-}) - \sum_{i=2}^{\infty}\frac{S_{s-}^iD_2^iF(s,S_{s-})}{i!B_s}H_{s-}^{(i)}}{B_s}\mathrm{d}B_s \\
& \quad + \int_t^T D_2F(s,S_{s-})\mathrm{d}S_s + \sum_{i=2}^{\infty}\int_t^T \frac{S_{s-}^iD_2^iF(s,S_{s-})}{i!B_s}\mathrm{d}H_{s-}^{(i)}
\end{aligned}
$$

In conclusion, we thus obtain that in order to hedge the claim X, we should have at time t a portfolio consisting out of: $D_2F(t,S_{t-})$ number of stock (this correspondence with the classical delta-hedge), $\frac{S_{t-}^iD_2^iF(t,S_{t-})}{i!B_t}$ number of the ith-power-jump assets and $\frac{F(s,S_{s-})-S_{s-}D_2F(s,S_{s-})-\sum_{i=2}^{\infty}\frac{S_{s-}^iD_2^iF(s,S_{s-})}{i!B_s}H_{s-}^{(i)}}{B_s}$ number of bonds. •

Remark 6 *Under the framework of the market enlarged with the orthonormalized power-jump assets one can prove a similar statement using Lemma 5 in [22].*

3.5 Degenerate Cases

3.5.1 The Black-Scholes Model

Suppose our risk-neutral dynamics of the stock-price are given by the Black-Scholes SDE

$$
\frac{\mathrm{d}S_t}{S_t} = (r - \frac{1}{2}\sigma^2)\mathrm{d}t + \mathrm{d}W_t, \qquad S_0 > 0,
$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion.

Note that in this case, the stock price process $S = \{S_t, t \geq 0\}$ is given by

$$
S_t = S_0\exp((r - \sigma^2/2)t + \sigma W_t),
$$

and that all processes $H^{(i)}$, $i = 2, \ldots$ are equal to zero. Hence it should be clear the market is already complete and that an enlargement is not necessary. Moreover the hedging portfolio is given by $\frac{F(s,S_s)-S_sD_2F(s,S_s)}{B_s}$ number of bonds and $D_2F(s,S_s)$ number of stocks.

3.5.2 The Poisson Case

Suppose our risk-neutral dynamics of the stock-price are given by

$$\frac{\mathrm{d}S_t}{S_{t-}} = (r-\lambda)\mathrm{d}t + \mathrm{d}N_t, \qquad S_0 > 0, \tag{9}$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter $\lambda > 0$.

Note that in this case, the stock price process $S = \{S_t, t \geq 0\}$ is given by

$$S_t = S_0 \exp((r-\lambda)t) 2^{N_t},$$

and that all compensated power-jump processes equal the compensated Poisson process:

$$Y_t^{(i)} = N_t - \lambda t, \qquad i \geq 2.$$

Consider a contingent claim with a payoff $F(T, S_T)$ only depending on the value of the stock price at maturity T. Note that in this very special case the payoff is thus also a function of the Poisson process at time T. We make this clear using the notation: $G(T, N_T) = F(T, S_T)$. Note that $G(T, N_T+1) = F(T, 2S_T)$.

We will make use of the following relation,

$$f(2x) - f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i!} \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x),$$

for functions f where the righthand-side is correctly defined.

First we note that

$$\begin{aligned}
&\sum_{i=2}^{\infty} \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i!} \\
&\quad = \quad F(s, 2S_{s-}) - F(s, S_{s-}) - S_{s-} D_2 F(s, S_{s-}) \\
&\quad = \quad G(s, N_{s-}+1) - G(s, N_{s-}) - S_{s-} D_2 F(s, S_{s-})
\end{aligned}$$

Hence

$$\begin{aligned}
&F(T, S_T) - F(t, S_t) \\
&\quad = \int_t^T \frac{F(s, S_{s-}) - S_{s-} D_2 F(s, S_{s-})}{B_s} \mathrm{d}B_s \\
&\qquad - \int_t^T \frac{(G(s, N_{s-}+1) - G(s, N_{s-}) - S_{s-} D_2 F(s, S_{s-}))(N_s - \lambda s)}{B_s} \mathrm{d}B_s \\
&\qquad + \int_t^T D_2 F(s, S_{s-}) \mathrm{d}S_s \\
&\qquad + \int_t^T B_s^{-1} (G(s, N_{s-}+1) - G(s, N_{s-}) - S_{s-} D_2 F(s, S_{s-})) \mathrm{d}B_s (N_s - \lambda s) \\
&\quad = \int_t^T \frac{F(s, S_{s-}) - S_{s-} D_2 F(s, S_{s-})}{B_s} \mathrm{d}B_s
\end{aligned}$$

$$
\begin{aligned}
& + \int_t^T D_2F(s,S_{s-})\mathrm{d}S_s \\
& + \int_t^T B_s^{-1}(G(s,N_{s-}+1)-G(s,N_{s-})-S_{s-}D_2F(s,S_{s-}))B_s\mathrm{d}(N_s-\lambda s) \\
= \; & \int_t^T \frac{F(s,S_{s-})-G(s,N_{s-}+1)+G(s,N_{s-})}{B_s}\mathrm{d}B_s \\
& + \int_t^T (G(s,N_{s-}+1)-G(s,N_{s-}))S_{s-}^{-1}\mathrm{d}S_s
\end{aligned}
$$

In conclusion, it should be clear the market is already complete and that an enlargement is not necessary. Moreover the hedging portfolio is given by $\frac{2G(s,S_{s-})-G(s,N_{s-}+1))}{B_s}$ number of bonds and $(G(s,N_{s-}+1)-G(s,N_{s-}))S_{s-}^{-1}$ number of stocks. This could also be directly derived using Itô's lemma for the Poisson process on the function G.

3.6 Pricing Formulas

Consider the value at time t of a contingent claim X with a payoff function $f(S_T)=F(T,S_T)$ only depending on the stock price at maturity:

$$F(t,S_t)=\exp(-r(T-t))E_Q[X|\mathcal{F}_t]=\exp(-r(T-t))E_Q[f(S_T)|\mathcal{F}_t].$$

Note that because we are working risk-neutrally that $a+b=r$. Then

$$
\begin{aligned}
& F(t,S_t)=\exp(-r(T-t))\times \\
& \quad E_Q\left[f\left(S_t\mathrm{e}^{c(W_T-W_t)+(L_T-L_t)+\left(r-\frac{c^2}{2}\right)(T-t)}\prod_{t<s\leq T}(1+\Delta L_s)\mathrm{e}^{-\Delta L_s}\right)|\mathcal{F}_t\right].
\end{aligned}
$$

Or

$$
\begin{aligned}
& F(t,x)=\exp(-r(T-t))\times \\
& \quad E_Q\left[f\left(x\mathrm{e}^{cW_{T-t}+L_{T-t}+\left(r-\frac{c^2}{2}\right)(T-t)}\prod_{0<s\leq T-t}(1+\Delta L_s)\mathrm{e}^{-\Delta L_s}\right)\right].
\end{aligned}
$$

If we introduce the function

$$F_{BS}(t,x)=\exp(-r(T-t))E_Q\left[f\left(x\exp\left(cW_{T-t}\left(r-\frac{c^2}{2}\right)(T-t)\right)\right)\right],$$

which gives the price of the option for the Black-Scholes model (with volatility c), we have

$$
\begin{aligned}
& F(t,x)= \\
& \quad E_Q\left[F_{BS}\left(t,x\mathrm{e}^{L_{T-t}}\prod_{0<s\leq T-t}(1+\Delta L_s)\mathrm{e}^{-\Delta L_s}\right)\right].
\end{aligned}
\tag{10}
$$

Remark 7 *Equation (10) and smoothness properties of F_{BS} can typically be used to prove that F satisfies the conditions mentioned in Theorem 5.*

Note that the calculation of the nth derivative with respect to x, as is needed in the formula for the number of nth power-jump assets in the replicating portfolio, is just

$$\begin{aligned} D_2^n F(t,x) &= \\ & E_Q[\mathrm{e}^{nL_{T-t}} \prod_{0<s\le T-t} (1+\Delta L_s)^n \mathrm{e}^{-n\Delta L_s} \times \\ & D_2^n F_{BS}\left(t, x\mathrm{e}^{L_{T-t}} \prod_{0<s\le T-t} (1+\Delta L_s)\mathrm{e}^{-\Delta L_s}\right)]. \end{aligned}$$

Recall that in case of the case of a classical European option, the $D_2^n F_{BS}$ are very simple. In case of the European call for example the first two derivatives are given in terms of the cumulative probability distribution function $\mathrm{N}(x)$ and the density function $\mathrm{n}(x)$ of a Standard Normal random variable by

$$D_2^1 F_{BS}(t,x) = \mathrm{N}(d_1) = \mathrm{N}\left(\frac{\log(x/(T-t)) + (r+c^2/2)(T-t)}{c\sqrt{T-t}}\right)$$

and

$$D_2^2 F_{BS}(t,x) = \mathrm{n}(d_1)/(xc\sqrt{T-t}),$$

which are also known as the *delta* and the *gamma* of the option.

4 Arbitrage

In this section we comment on the reverse question in a particular example. More precisely, we assume our market is already enlarged with the power-jump assets. So, we have chosen the constants $a^{(i)}$, $i = 2, 3, \dots$ and trade is allowed in the bond, the stock and the power-jump assets with price process $H_t^{(i)} = \exp(rt)(X_t^{(i)} - a^{(i)}t)$, $i = 2, 3, \dots$.

The precise choice of the constants $a^{(i)}$ is delicate and will be discussed in detail below. Moreover, we will show the need of the factor $\exp(rt)$ and these constants. Essentially, these are needed to prevent arbitrage opportunities. The constants $a^{(i)}$ can for example not be set to zero, since then in case of $r = 0$, all $H_t^{(i)}$ with even i are strictly increasing and starting at zero. This clearly, will lead to arbitrage opportunities if trade is allowed in these objects.

We investigate whether this enlargement leads to arbitrage or not. No-arbitrage is implied by the existence of an equivalent martingale measure under which all discounted assets in the market are martingales. This question is related with the moment problem. We only solve the question partially: (i) We only consider the case of a model driven by a Brownian Motion and a finite number of independent Poisson process, (ii) moreover we only give sufficient conditions to ensure that there exist an equivalent martingale measure (and hence the market is arbitrage free): in continuous time the existence of an equivalent martingale measure is a sufficient but not a necessary condition to ensure no-arbitrage. The problem in its full generality seems to be very hard and challenging.

Let us start with assuming that our Lévy processes has a Brownian component, i.e. $c \neq 0$, and that trade is allowed in the assets with price processes $H^{(i)} = \{H_t^{(i)}, t \geq 0\}$.

4.1 Equivalent Martingale Measures

In this section we will describe the many measures, equivalent to the canonical (real world) measure under which the discounted stock price process is a martingale and under which Z remains a Lévy process. More precisely, we characterize all *structure preserving* P-equivalent martingale measures Q under which Z remains a Lévy process and the process $\tilde{S} = \{\tilde{S}_t = \exp(-rt)S_t, t \geq 0\}$ is a $\{\mathcal{F}_t\}$-martingale, where $\mathcal{F}_t = \sigma(S_u : 0 \leq u \leq t) \vee \mathcal{N}$ is the natural filtration generated by the stock price process completed with the P-null sets. Since we are considering a market with finite horizon T then $0 \leq t \leq T$ and locally equivalence for any t will be the same as equivalence.

We have the following result (see [25, Theorem 33.1]):

Theorem 8 *Let Z be a Lévy process with Lévy triplet $[\alpha, c^2, \nu(dx)]$ under some probability measure P. Then the following two conditions are equivalent.*

- *(a) There is a probability measure $\tilde{P}$ locally equivalent to P for any $t \geq 0$, such that Z is a $\tilde{P}$-Lévy process with triplet $[\tilde{\alpha}, \tilde{c}^2, \tilde{\nu}(dx)]$.*
- *(b) All of the following conditions hold.*
  - *(i) $\tilde{\nu}(dx) = H(x)\nu(dx)$ for some Borel function $H : \mathbb{R} \rightarrow (0, \infty)$.*
  - *(ii) $\tilde{\alpha} = \alpha + \int_{-1}^{1} x(H(x) - 1)\nu(dx) + Gc$ for some $G \in \mathbb{R}$.*
  - *(iii) $\tilde{c} = c$.*
  - *(iv) $\int_{-\infty}^{\infty} (1 - \sqrt{H(x)})^2 \nu(dx) < \infty$.*

Remark 9 *We note that if one does not impose that Z should remain a Lévy process under $\tilde{P}$, then G and H are not necessarily independent of time.*

For a (locally) P-equivalent martingale measure $\tilde{P}$, we have

$$\left. \frac{\mathrm{d}\tilde{P}}{\mathrm{d}P} \right|_{\mathcal{F}_T} = K_T,$$

where

$$\begin{aligned} K_T &= \exp\left(GW_T - \frac{1}{2}G^2 T + \int_0^T \int_{-\infty}^{+\infty} (H(x) - 1) M(\mathrm{d}s, \mathrm{d}x) \right) \\ &\quad \times \prod_{0 < s \leq T} H(\Delta X_s) \exp(1 - H(\Delta X_s)). \end{aligned}$$

for some G and $H(x) > 0$, with $H(0) = 1$, for which $E_P[K_T] = 1$. Moreover, under $\tilde{P}$, the process

$$\tilde{W}_t = W_t - Gt$$

is a Brownian motion and the process X is a quadratic pure jump Lévy process with Lévy measure given by

$$\tilde{\nu}(\mathrm{d}x) = H(x)\nu(\mathrm{d}x)$$

and predictable part given by

$$\tilde{a}t = E_{\tilde{P}}[X_t] = (a + \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x))t.$$

See [25, Theorem 33.2].

We now want to find an equivalent martingale measure Q under which the discounted price process $\tilde{S}$ is a martingale. By the above theorem, under such a Q, X has Doob-Meyer decomposition

$$X_t = \tilde{L}_t + (a + \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x))t,$$

where $\tilde{L} = \{\tilde{L}_t, t \geq 0\}$ is a Q-martingale. Noting that $\Delta L_t = \Delta\tilde{L}_t$, we have

$$\begin{aligned} \tilde{S}_t &= S_0 \exp\left(c\tilde{W}_t + \tilde{L}_t + \left(a + b - r + cG - \frac{c^2}{2}\right)t\right) \times \\ &\quad \exp\left(t\int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x)\right) \prod_{0<s\leq t}(1 + \Delta\tilde{L}_s)\exp(-\Delta\tilde{L}_s). \end{aligned}$$

A necessary and sufficient condition for $\tilde{S}$ to be a Q-martingale is the existence of G and $H(x)$, with $\int_{-\infty}^{\infty}(1 - \sqrt{H(x)})^2\nu(\mathrm{d}x) < \infty$ for which the process K as is a positive martingale and such that

$$cG + a + b - r + \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x) = 0. \tag{11}$$

Remark 10 *We remark (see e.g. [15] or [7]), that if there exists a (non-structure preserving) locally equivalent martingale measure $\tilde{P}_1$ under which Z is not anymore a Lévy process, there exists always a (structure preserving) locally equivalent martingale measure $\tilde{P}_2$ under which Z is a Lévy process.*

In order to have no arbitrage, there should be an equivalent martingale measure Q not only making the discounted stock price process a martingale but also making the discounted $H^{(i)}$'s martingales. More precisely, the condition that the discounted stock price must be a martingale again comes down to the existence of G and $H(x)$ such at condition (11) holds. However, now also the discounted $H^{(i)}$'s, i.e.

$$\tilde{H}_t^{(i)} = X_t^{(i)} - a^{(i)}t,$$

need to be martingales. Using (4) and together with the fact that the Lévy measure of X under Q is given by $H(x)\nu(\mathrm{d}x)$, this comes down to

$$\int_{-\infty}^{+\infty} x^i H(x)\nu(\mathrm{d}x) = a^{(i)}, \quad i = 2, 3, 4, ... \tag{12}$$

The question thus now is, do there exists G and $H(x)$ such that (11) and (12) hold simultaneously. This question is very related with the moment problem: given a series of numbers $\{\mu_n\}$ find necessary and sufficient conditions in order that there exists a measure with μ_n as nth moment. The measure should satisfy some imposed conditions like being supported on a predefined set, being bounded or absolutely continuous with respect to another measure, etc.

When a moment problem has a solution and the corresponding (positive-definite) moment functional is determinate (i.e. unique), the moment problem is said to be *determined.* The question of when of moment functional is determinate is a difficult one. A partially result is that if the moment functional has a bounded supporting set, it is determinate. So if the moment problem has a solution whose spectrum is bounded, then it will be a determined moment problem. For the general question of determinacy see [27] or [1].

4.2 Special Cases

4.2.1 Brownian Motion plus a Poisson Process

Suppose

$$Z_t = cW_t + N_t,$$

where $c \neq 0$, $W = \{W_t, t \geq 0\}$ a standard Brownian Motion and $N = \{N_t, t \geq 0\}$ is an independent Poisson process with intensity $a > 0$. Then $X_t = N_t$ and

$$H_t^{(i)} = H_t^{(2)} = \exp(rt)(N_t - a^{(2)}t), i = 2, 3, \dots.$$

So we enlarge the market with only one object, namely the asset following the price process $H_t^{(2)} = \exp(rt)(N_t - a^{(2)}t)$. In order that an equivalent martingale measure Q exists, we must have the existence of a G and H, such that

$$\begin{aligned} \int_{-\infty}^{+\infty} x(H(x)-1)\nu(\mathrm{d}x) &= r - cG - a - b \\ \int_{-\infty}^{+\infty} x^2 H(x)\nu(\mathrm{d}x) &= a^{(2)} \end{aligned}$$

Since H is only relevant at the support of ν, i.e. just the point 1 in this case, these equations reduce to

$$\begin{aligned} -\frac{aH(1) - r + b}{c} &= G \\ aH(1) &= a^{(2)} \end{aligned}$$

In order to give rice to an equivalent martingale measure $H(1) = a^{(2)} > 0$.

4.2.2 Brownian Motion plus a finite number of Poisson Processes

Suppose

$$Z_t = cW_t + \sum_{j=1}^{n} c_j N_{j,t},$$

where $c \neq 0$, $W = \{W_t, t \geq 0\}$ a standard Brownian Motion and $N_j = \{N_{j,t}, t \geq 0\}$ are independent Poisson process with intensity $a_j > 0$. The constants c_j, $j = 1, \ldots, n$ are assumed to be all different from each other and non-zero. Then $X_t = \sum_{j=1}^{n} c_j N_{j,t}$ and $E[X_1] = \sum_{j=1}^{n} c_j a_j = a$

$$H_t^{(i)} = \exp(rt)(\sum_{j=1}^{n} c_j^i N_{j,t} - a^{(i)}t), i = 2, 3, \ldots.$$

It is not that hard to see that $H_t^{(i)}$, for $i > n+1$ can be written as a linear combination of the $H_t^{(i)}$, $i = 2, \ldots, n+1$ (see [18]). In this case we enlarge the market with only n object, namely the asset following the price process $H_t^{(i)}$, $i = 2, \ldots, n+1$. In order that an equivalent martingale measure Q exists, we must have the existence of a G and H, such that

$$\begin{aligned} \int_{-\infty}^{+\infty} x(H(x) - 1)\nu(\mathrm{d}x) &= r - cG - a - b \\ \int_{-\infty}^{+\infty} x^i H(x)\nu(\mathrm{d}x) &= a^{(i)}, \quad i = 2, \ldots, n+1. \end{aligned}$$

The support of H will now be the set $\{c_2, \ldots, c_{n+1}\}$ and the above equations reduce to

$$\begin{aligned} \sum_{j=1}^{n} c_j H(c_j) a_j &= r - cG - b \\ \sum_{j=1}^{n} c_j^i H(c_j) a_j &= a^{(i)}, \quad i = 2, \ldots, n+1. \end{aligned}$$

There exists an equivalent martingale measure if the following system of equations for $H(c_j)$, $j = 1, \ldots, n$ has a positive solution, i.e. $H(c_j) > 0$, $j = 1, \ldots, n$.

$$\left[\begin{array}{cccc} c_1^2 a_1 & c_2^2 a_2 & \ldots & c_n^2 a_n \\ c_1^3 a_1 & c_2^3 a_2 & \ldots & c_n^3 a_n \\ \ldots & \ldots & \ldots & \ldots \\ c_1^{n+1} a_1 & c_2^{n+1} a_2 & \ldots & c_n^{n+1} a_n \end{array}\right] \times \left[\begin{array}{c} H(c_1) \\ H(c_2) \\ \ldots \\ H(c_n) \end{array}\right] = \left[\begin{array}{c} a^{(2)} \\ a^{(3)} \\ \ldots \\ a^{(n+1)} \end{array}\right].$$

The existence of a positive solution $H(c_j)$, $j = 1, \ldots, n$ can be translated into the condition

$$C^{-1} \cdot a' > 0, \tag{13}$$

where C^{-1} is the inverse of the Vandermonde matrix

$$C = \begin{bmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_n \\ \dots & \dots & \dots & \dots \\ c_1^{n-1} & c_2^{n-1} & \dots & c_n^{n-1} \end{bmatrix}$$

and a' is the transpose of $[a^{(2)} \dots a^{(n+1)}]$. Note that if all c_i's are different from each other (as we assumed above) that $\det C \neq 0$.

For the calculation of the inverse of Vandermonde matrices see [16] or [19]. For other applications of Vandermonde matrices in finance see [20].

Remark 11 *In case $n = 2$, assuming $c_2 > c_1$ condition (13) can be written as $a^{(2)}c_2 > a^{(3)} > a^{(2)}c_1$.*

Remark 12 *Note that the condition (13) is only a sufficient condition to have an arbitrage-free market.*

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